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# Geometric Phase for Optical Free Induction Decay

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**Abstract** Geometric phase is investigated for optical free induction decay with a modified Bloch equation by establishing in connecting density matrices with nonunit vector ray in a complex projective Hilbert space. Under the limiting of pure state, our approach may give out the Berry phase and Aharonov and Anandan one. Furthermore, by comparing our approach with the kinematic one, we find that, after a suitable modification to the kinematic approach, both differences are very small for the Berry phase of mixed states in the optical free induction decay under the case of quasicyclic evolution.

The relations between the population inversion and geometric phase and between the normalized intensity of the heterodyne beat signal and the geometric one are analyzed in process of optical free induction decay. We find that the population inversion and normalized intensity of signal field are a linear function of the geometric phase, which may be helpful to keep temporal storage of quantum information in a long distance quantum communication by using geometric phase.

**Keywords** Geometric phase · Mixed state · Optical free induction decay

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## 1 Introduction

The wave function of a quantum system retains a memory of its motion in terms of a geometric phase [1–5] when it undergoes a closed evolution in parameter space. For a pure state, the geometric phase essentially arises as an effect of parallel transport in the Poincaré representation of the manifold of pure states. This phase factor can be measured in principle by interfering the wave function which has undergone the above evolution with another coherent wave function that did not evolve, which enable one to discern whether or not the system has undergone an evolution by the geometric phase. The geometric phase for the pure state has been observed in spin $1/2$ systems through nuclear-magnetic-resonance (NMR) experiments [6] and with polarized photons using interferometers (PPI) [7, 8] and photon echoes [9].

Application of the geometric phases in quantum computation [10–13] has motivated their studies under more realistic situations [14–29]. In a real system, noise and decoherence are big problems. The process limits the ability to maintain pure quantum states in quantum information process. Differently from the pure state, the mixed state for the open system is always written in many different ways as a probabilistic mixture of distinct but not necessarily orthogonal pure states. Thus, the density matrix was introduced as a way of describing the quantum open system and the state of the open system is not completely known. Up to now, the definition of the geometric phase for the open system is still a controversial issue. There have been many proposals tackling the problem from different generalizations of the parallel transport condition [24]. However, a general belief is that the Berry phases are geometric in their nature, i.e., proportional to the area spanned in parameter space. Therefore, the geometric phase for the open system should be expressed here in terms of geometric structures on a complex projective Hilbert space [27]. Thus, it is interesting in comparison with these different approaches to the geometric phase of open system. Differently from the previous many works about the geometric phase within Bloch equation for the open system, especially, we consider here a modified Bloch equation for optical free induction decay.

On the other hand, it is known that the current quantum computation presents a wide range of challenges to quantum information [30–32], particularly the search for devices of quantum memory that allow temporal storage of quantum information in a long distance quantum communication. The photon echo approach may be one of the available methods for quantum memory [33–36]. Implement of the efficient quantum memory for single photon and intensive specifically quantum light fields has attracting considerable attention in the last few years [37–40]. In addition to its fundamental importance for quantum processing and communications, the quantum memory promises a new tool for more precise experimental investigation of atom-photon interactions and the effective generation of new quantum states of light [41, 42]. Moreover, the photon echo experiments are potentially sensitive detectors of the geometric phase [9, 43].

In the stimulated echo configuration, three excitation pulses are applied at the different times, then an echo is observed. It is known that atoms or molecules in different environments have different resonant frequencies so as to lead to the inhomogeneous broadening of a spectral line, which correspond to different processes, such as optical nutation, optical free induction decay and echo. It is therefore extremely important to understand all aspects of the photon echo, especially for the optical free induction decay where in a photon echo based on quantum memory, the retrieved signal decay will leads to the limited duration of the stored signal.

In addition, it is known that the geometric phase has observable consequences in quantum evolution. Thus it is interesting to study the relations between the geometric phase and the physical quantities in processes of the optical free induction decay.

The organization of the paper is as follows. In Sect. 2, we briefly review about the optical free induction decay. The geometric phase in the process of optical free induction is investigated in Sect. 3. In Sect. 4, furthermore, we discuss the relations between the geometric phase and the population inversion and between the geometric phase and the heterodyne beat signal. In Sect. 5, we compare our results with the kinematic one. At last, the conclusions are given in Sect. 6.

## 2 Optical Free Induction Decay

A resonantly excited atom or molecule acquires a polarization, oscillating at the natural frequency, which radiates and eventually decays away. This basis emission process is called as the free induction decay [44–49].

Let us consider a set of overlapping optical transitions that are simultaneously excited in steady state for all molecules by a laser field with the frequency $\omega$, wave vector $k$ and amplitude $E_0$,

$$E_x(z,t) = E_0\cos(\omega t - kz) = \frac{E_0}{2}[\exp[i(\omega t - kz)] + \exp[-i(\omega t - kz)]], \tag{1}$$

which is polarized along the $x$ direction that is taken to be parallel to the electric dipole transition moment, and propagates in the $z$ direction. Under the case of this field only coupling with two-level system $|a\rangle$ and $|b\rangle$, the time derivatives of the elements of the density matrix may be written as

$$\dot{\rho}_{aa}(z,v_z,t) = -\gamma_a\rho_{aa}(z,v_z,t) - \frac{i}{\hbar}(\rho_{ba}(z,v_z,t) - \rho_{ab}(z,v_z,t))V_{ab}, \tag{2}$$

$$\dot{\rho}_{bb}(z,v_z,t) = -\gamma_b\rho_{bb}(z,v_z,t) + \frac{i}{\hbar}(\rho_{ba}(z,v_z,t) - \rho_{ab}(z,v_z,t))V_{ab}, \tag{3}$$

$$\dot{\rho}_{ab}(z,v_z,t) = -(i\omega_{ab} + \gamma)\rho_{ab}(z,v_z,t) + \frac{i}{\hbar}(\rho_{aa}(z,v_z,t) - \rho_{bb}(z,v_z,t))V_{ab}, \tag{4}$$

$$\rho_{ba}(z,v_z,t) = \rho^*_{ab}(z,v_z,t), \tag{5}$$

where $v_z$ is velocities of molecules in the medium and $\hbar\omega_{ab} = \hbar\omega_a - \hbar\omega_b$ is an energy difference between energy levels $|a\rangle$ and $|b\rangle$ and $V_{ab} = -DE_x(z,t)$ is an interacting matrix element between the molecule dipole-moment $D$ and the laser field $E_x$. The longitudinal and transverse relaxation times $T_1$ and $T_2$ are introduced directly by $T_1 = 1/\gamma_a = 1/\gamma_b$ and $T_2 = 1/\gamma$, where $\gamma_a$, $\gamma_b$ and $\gamma$ are radiative decay rates. In order to remove the factors with varying of time in the nondiagonal elements, furthermore, the transformations, such as $\rho_{ab} = \tilde{\rho}_{ab}\exp[-i(\omega t - kz)]$ and $\rho_{ba} = \tilde{\rho}_{ba}\exp[i(\omega t - kz)]$, are introduced so that $\tilde{\rho}_{ab}$ and $\tilde{\rho}_{ba}$ are slowly varying functions of time. by denoting $\chi = DE_0/\hbar$, the equations of density operators under the rotation wave approximation are rewritten as,

$$\dot{\tilde{\rho}}_{ab} + \dot{\tilde{\rho}}_{ba} = -i\Delta(\tilde{\rho}_{ab} - \tilde{\rho}_{ba}) - \frac{\tilde{\rho}_{ab} + \tilde{\rho}_{ba}}{T_2}, \tag{6}$$

$$i(\dot{\tilde{\rho}}_{ab} - \dot{\tilde{\rho}}_{ba}) = \Delta(\tilde{\rho}_{ab} + \tilde{\rho}_{ba}) + \chi(\rho_{aa} - \rho_{bb}) - \frac{i(\tilde{\rho}_{ab} - \tilde{\rho}_{ba})}{T_2}, \tag{7}$$

$$\dot{\rho}_{aa} - \dot{\rho}_{bb} = -i\chi(\tilde{\rho}_{ab} - \tilde{\rho}_{ba}) - \frac{\rho_{aa} - \rho_{bb}}{T_1} + \frac{\rho^0_{aa} - \rho^0_{bb}}{T_1}, \tag{8}$$

where $\rho^0_{aa}$ and $\rho^0_{bb}$ are the probabilities, of the system that is found in the energy levels $a$ and $b$, at the equilibrium values of $\rho_{aa}$ and $\rho_{bb}$ in the absence of the laser beam, respectively. While $\Delta = -\omega + kv_z + \omega_{ab}$ is a harmonic parameter of molecules and $kv_z$ is the Doppler shift.

From (6)–(8), it is convenient to introduce Bloch vectors by $u = \tilde{\rho}_{ab} + \tilde{\rho}_{ba}$, $v = i(\tilde{\rho}_{ab} - \tilde{\rho}_{ba})$ and $w = \rho_{aa} - \rho_{bb}$, where $w$ is also an importantly physical quantity to describe the population inversion in the process of optical free induction decay.

During $t \leq 0$, for the optical free induction, the steady state $\dot{u} = \dot{v} = \dot{w} = 0$ is prepared, which leads to the initial conditions as $u(0) = -\Delta\chi w^0/\sigma$, $v(0) = \chi w^0/(T_2\sigma)$ and $w(0) = w^0[1 - \chi^2 T_1/(T_2\sigma)]$, where $w^0 = \rho^0_{aa} - \rho^0_{bb}$, $\sigma = \chi^2 T_1/T_2 + \Delta^2 + 1/T_2^2$.

When the Stark field is switched during $t \geq 0$, the molecules in the medium are far out of resonance. The polarization radiates are at the new frequency,

$$\Delta \to \Delta' = \Delta + \delta\omega_0, \tag{9}$$

which makes that the resonance condition is not satisfied between the laser filed and medium so that one may not consider the terms including $\chi$ in (6)–(8) because of the Stark shift of frequency $\delta\omega_0$. Thus the modified Bloch equation becomes

$$\dot{u} = -\Delta' v - u/T_2, \tag{10}$$

$$\dot{v} = \Delta' u - v/T_2, \tag{11}$$

$$\dot{w} = -(w - w^0)/T_1, \tag{12}$$

which is different from the usual Bloch equation, where a constant term $w^0/T_1$ is included in (12). The solutions of (10)–(12) are direct and we have

$$u(t) = [u(0)\cos\Delta' t - v(0)\sin\Delta' t]\exp[-t/T_2], \tag{13}$$

$$v(t) = [u(0)\sin\Delta' t + v(0)\cos\Delta' t]\exp[-t/T_2], \tag{14}$$

$$w(t) = w^0 + [w(0) - w^0]\exp[-t/T_1], \tag{15}$$

where the Doppler averaged density matrix element is given by $\langle u\rangle = 0$, i.e.,

$$\begin{aligned}\langle\tilde{\rho}_{ab}\rangle &\approx \frac{-i}{2\sqrt{\pi}kv_p}\exp\left[-\left(\frac{\Delta_c}{kv_p}\right)^2\right]\int_{-\infty}^{+\infty}[u(0)\sin\Delta' t + v(0)\cos\Delta' t]\exp[-t/T_2]d\Delta \\ &= \frac{-i\sqrt{\pi}}{2kv_p}\exp\left[-\left(\frac{\Delta_c}{kv_p}\right)^2\right]\chi w^0\exp\left[-\frac{t}{T_2}(\sqrt{\chi^2 T_1 T_2 + 1} + 1)\right] \\ &\quad\times\left(\frac{1}{\sqrt{\chi^2 T_1 T_2 + 1}} - 1\right)\cos\delta\omega_0 t,\end{aligned} \tag{16}$$

where the Doppler factor $\exp[-(\frac{\Delta_c}{kv_p})^2]$ is assumed as a slowly varying function of time and taken outside the integral. While $v_p$ is a thermal velocity, $\Delta_c$ is a Doppler central frequency of Doppler broadened transition.

Thus, the signal field is given by

$$E_{ab}(L,t) = 2\pi ikN\mu_{ab}\langle\tilde{\rho}_{ab}\rangle = \frac{\pi^{3/2}}{v_p} NDL\chi w^0 \exp\left[-\frac{t}{T_2}(\sqrt{\chi^2 T_1 T_2 + 1} + 1)\right] \times \left(\frac{1}{\sqrt{\chi^2 T_1 T_2 + 1}} - 1\right)\cos\delta\omega_0 t \exp\left[-\left(\frac{\Delta_c}{k v_p}\right)^2\right], \tag{17}$$

where $N$ is the number of molecules, $L$ is length between laser source and detector and $\mu_{ab} = DL$.

The total field striking the detector is plus and the heterodyne beat signal is obtained by the cross product, i.e.,

$$I_t(t) \approx 2E_0 E_{ab}(L,t) = \frac{2\pi^{3/2}}{v_p\hbar} ND^2 LE_0^2 w^0 \exp\left[-\left(\frac{\Delta_c}{k v_p}\right)^2\right] \times \exp\left[-\frac{t}{T_2}(\sqrt{\chi^2 T_1 T_2 + 1} + 1)\right]\left(\frac{1}{\sqrt{\chi^2 T_1 T_2 + 1}} - 1\right)\cos\delta\omega_0 t. \tag{18}$$

Thus the normalized intensity may be expressed as

$$\frac{I_t(t)}{I_t(0)} = \exp\left[-\frac{t}{T_2}(\sqrt{\chi^2 T_1 T_2 + 1} + 1)\right]\cos\delta\omega_0 t. \tag{19}$$

The normalized intensity as a function of the evolving time $t$ is shown at Fig. 1 for the different frequency shifts. From Fig. 1, we see that the intensity of heterodyne beat signal oscillates with the evolving time and the oscillating values are at the range of $[-1, 1]$ because

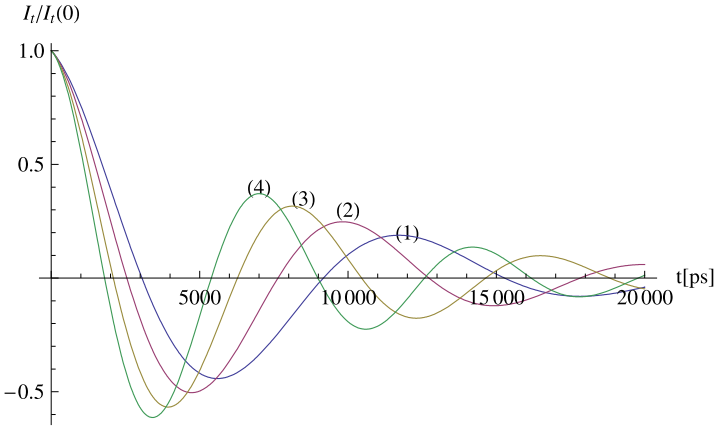


**Fig. 1** The normalized intensity of the heterodyne beat signal $I_t(t)/I_t(0)$ is as a function of the evolving time $t$ with the parameters $T_1 = 0.97 \times 10^{-9}$ s, $T_2 = 40 \times 10^{-9}$ s, $\chi = 2.1 \times 10^8$ and $w^0 = 0.1$ for the different frequency shifts (*1*) $\delta\omega_0 = 5.14 \times 10^8$ rad/s, (*2*) $\delta\omega_0 = 6.17 \times 10^8$ rad/s, (*3*) $\delta\omega_0 = 7.51 \times 10^8$ rad/s and (*4*) $\delta\omega_0 = 8.74 \times 10^8$ rad/s, respectively. The results show that the oscillating normalized intensity decays with the evolving time

the heterodyne beat signal is a cross product. Moreover, the heterodyne beat signal decays eventually away with increasing of the evolving time.

## 3 Geometric Phase for Optical Free Induction Decay

Recently, the searching for devices of quantum memory has motivated our studies to analyze the geometric phase for optical free induction decay because the photon echo approach may be one of the available methods [33–36], where the optical free induction decay is improtant in process of photon echo.

In the process of the optical free induction decay, the density matrices were introduced as a way to describe the interaction between the molecules or atoms and the laser field, where a state for the open system may be described by the modified Bloch equations (10)–(12).

Thus, we firstly expand Refs. [27–29] to include the modified Bloch equation, i.e., the geometric phase is formulated in terms of geometric structures on a complex projective Hilbert space so that the corresponding Berry phases for the mixed state are proportional to the area spanned in parameter space.

In order to obtain the geometric phase, firstly, the Bloch vectors are parameterized by the radius $r$ and two azimuthal angles $\alpha$ and $\beta$ of the Bloch sphere [27–29], i.e.,

$$\begin{aligned} r^2(t) &= u^2(t) + v^2(t) + w^2(t) \\ &= [u(0)\cos\Delta' t - v(0)\sin\Delta' t]^2 \exp(-2t/T_2) \\ &\quad - [u(0)\sin\Delta' t + v(0)\cos\Delta' t]^2 \exp(-2t/T_2) \\ &\quad + [w^0 + (w(0) - w^0)\exp(-t/T_1)]^2, \end{aligned} \tag{20}$$

which include, differently from the usual open system within the Bloch equation, the contribution from the constant term of the modified Bloch equation,

$$\alpha(t) = \cos^{-1}\frac{w}{r} = \cos^{-1}\frac{w^0 + (w(0) - w^0)\exp(-t/T_1)}{r}, \tag{21}$$

and

$$\beta(t) = \tan^{-1}\frac{v}{u} = \tan^{-1}\frac{u(0)\sin\Delta' t + v(0)\cos\Delta' t}{u(0)\cos\Delta' t - v(0)\sin\Delta' t}. \tag{22}$$

It is known that, when $r(t) = (u^2(t) + v^2(t) + w^2(t))^{1/2} = 1$, the Bloch vectors $(u(t), v(t), w(t))$ are points on this unit Poincaré sphere, which implies that the physical system is in a pure state at any evolving time.

For $r(t) = (u^2(t) + v^2(t) + w^2(t))^{1/2} < 1$, on the other hand, the Bloch vectors $(u(t), v(t), w(t))$ are interior points of this unit Poincaré sphere. It is known that the mixed states identify with the interior points of this sphere. From (20), it is proved that the Bloch radius is smaller than one in process of the optical free induction decay with the modification Bloch equation. Therefore, our physical system considered here is in the mixed state at all evolving time. Thus it is different from the open system within the Bloch equation, where the initial state is usually in the pure state.

By renormalizing for the Bloch vectors, the interior points in the unit Poincaré sphere may be mapped onto field amplitudes as the mixed states $|\psi\rangle$ as pointed out by one of the

authors [27], which may be expressed as

$$|\psi(t)\rangle = \begin{pmatrix} \sqrt{r}\cos\frac{\alpha(t)}{2} \\ \sqrt{r}\exp[i\beta(t)]\sin\frac{\alpha(t)}{2} \end{pmatrix}, \tag{23}$$

which includes the effect of optical free induction decay. Therefore, it is a dressed spin $-1/2$ state with nonunit vector in the complex projective Hilbert space.

It was known that the nonunit vector ray (23) in the projective Hilbert space is one-to-one correspondence with the Bloch vectors (13)–(15) [27]. Therefore, the approach from (23) to the geometric phase may be unique.

According to (23), the total phase of physical system is given by

$$\arg\langle\psi(t_0)|\psi(t)\rangle = \tan^{-1}\frac{\sin(\beta(t)-\beta(t_0))\sin\frac{\alpha(t_0)}{2}\sin\frac{\alpha(t)}{2}}{\cos\frac{\alpha(t_0)}{2}\cos\frac{\alpha(t)}{2}+\cos(\beta(t)-\beta(t_0))\sin\frac{\alpha(t_0)}{2}\sin\frac{\alpha(t)}{2}}, \tag{24}$$

and the geometric phase, expressed by the state vector in the complex projective Hilbert space, is obtained by [27–29],

$$\begin{aligned}\gamma_g &= \arg\langle\psi(t_0)|\psi(t)\rangle - \Im\int_{t_0}^{t}\frac{\langle\psi(t)|d|\psi(t)\rangle}{\langle\psi(t)|\psi(t)\rangle} \\ &= \arg\langle\psi(t_0)|\psi(t)\rangle - \frac{1}{2}\int_{t_0}^{t}(1-\cos\alpha(t))d\beta(t),\end{aligned} \tag{25}$$

which is a gauge and reparameterized invariance [50]. It is necessary to point out that (25) is a generalization for the Pantcharatnam formula in the pure state and therefore is called as the Pantcharatnam phase of the mixed state. It was proved that (25) was in agreement with the result directly from nounitary evolution, where the geometric phase is expressed by the density matrix [27, 50]. It is interesting to note that, furthermore, under the transformation, $|\overline{\psi}(t)\rangle = \exp(-i\arg\langle\psi(t_0)|\psi(t)\rangle)|\psi(t)\rangle$, the geometric phase (25) may be rewritten as

$$\gamma_g^A = -\Im\int_{t_0}^{t}\frac{\langle\overline{\psi}(t)|d|\overline{\psi}(t)\rangle}{\langle\overline{\psi}(t)|\overline{\psi}(t)\rangle}, \tag{26}$$

which is a generalization of the Aharonov and Anandan phase of the pure state with the condition of $\langle\overline{\psi}(t)|\overline{\psi}(t)\rangle = \langle\psi(t)|\psi(t)\rangle = 1$. Therefore, $\gamma_g^A$ is called as the Aharonov and Anandan phase of mixed state.

In process of the optical free induction decay, the system no longer undergoes a cyclic evolution, where the exponent decay factors are included in the density matrices, Bloch vectors and nonunit state vectors. When it is isolated from the environment, however, the evolving time $T = \frac{2\pi}{\Delta'}$ in the quasicyclic process may be regarded as its cyclicity so that the total phase in (24) is equal to $2\pi$, which is not important and may be dropped off in quantum computation. Thus, the geometric phase under the quasicyclic process may be expressed as

$$\begin{aligned}\gamma_g^B &= -\Im\int_0^T\frac{\langle\psi(t)|d|\psi(t)\rangle}{\langle\psi(t)|\psi(t)\rangle} \\ &= -\frac{1}{2}\int_0^T(1-\cos\alpha(t))d\beta(t),\end{aligned} \tag{27}$$

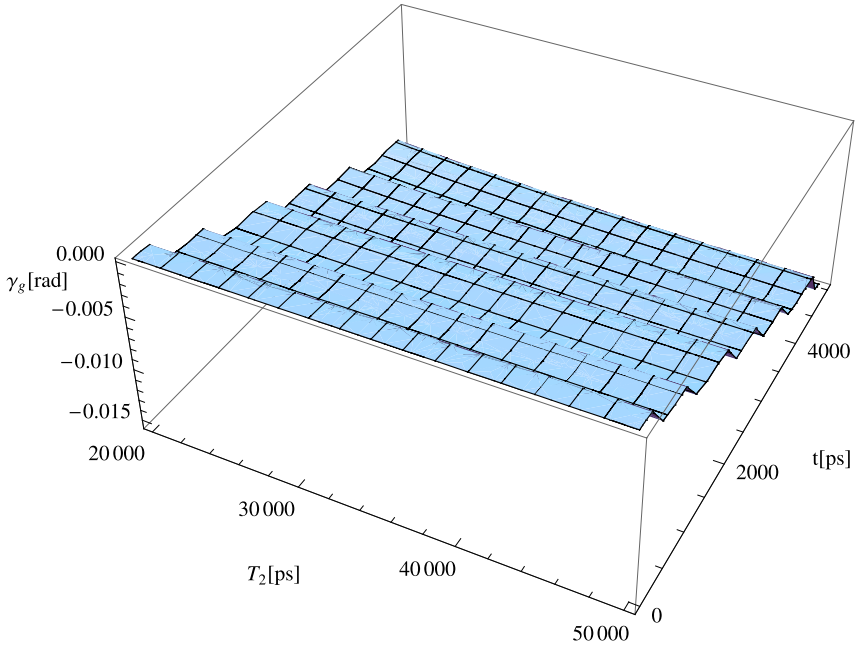


**Fig. 2** Geometric phase in optical free induction decay is as a function of the evolving and transverse relaxation times $T_2$ according to the relation $T_1 = T_2/(\gamma_h T_2 - 1)$ with $\gamma_h = 10^9/\text{s}$ for the parameters $\chi = 2.1 \times 10^8$, $w^0 = 0.1$ and $\Delta' = 3.6 \times 10^9$ rad/s, which oscillates and decreases with the evolving time

which is a generalized expression of the Berry phase of pure state and therefore is called the Berry phase of mixed state.

It is obvious that $\gamma_g^B$ is an area in parameter space under the approximation of the quasicyclic evolution [29] and is accumulated at region of $t \in [0, T]$ in a quasicyclic evolution. While $\gamma_g$ is a function of time associated with an evolution of a quantum open system [29, 51].

From (20)–(22) and (24)–(27), we see that, though the geometric phases has the same forms with the ones from the Bloch equation, they include the effect of constant term beyond the Bloch equation. The effect lost away within the Bloch equation.

The geometric (Pantcharatnam) phase as functions of the evolving time $t$ and transverse relaxation one $T_2$ for the relation $T_1 = T_2/(\gamma_h T_2 - 1)$ with $\gamma_h = 10^9/\text{s}$ is shown at Fig. 2. We find that the Pantcharatnam phase oscillates and decreases with the evolving time $t$. It is obvious that the oscillations are from the total phase (see Fig. 3). Comparing Fig. 1 with Fig. 2, we see that the phase appears to reflect the oscillation and decay properties of normalized intensity of the heterodyne beat signal $I_t(t)/I_t(0)$, the Differently from the Pantcharatnam phase, the oscillation disappears for the Berry phase (see Figs. 4–6) because it is accumulated at the region $t \in [0, 2\pi/\Delta']$. Moreover, we find that the Berry phase depends strongly on the relaxation times. It is an increasing function of the longitudinal relaxation time $T_1$ for a given $T_2$ (see Figs. 4 and 5). In contrast, it is a decreasing function of the transverse relaxation time $T_2$ for a given $T_1$ (see Figs. 5 and 6). In addition, the Berry phases are different for the different regions of evolving time (see Fig. 6) because the system does not exactly undergo a cyclic evolution, where the exponent decay factors are included.

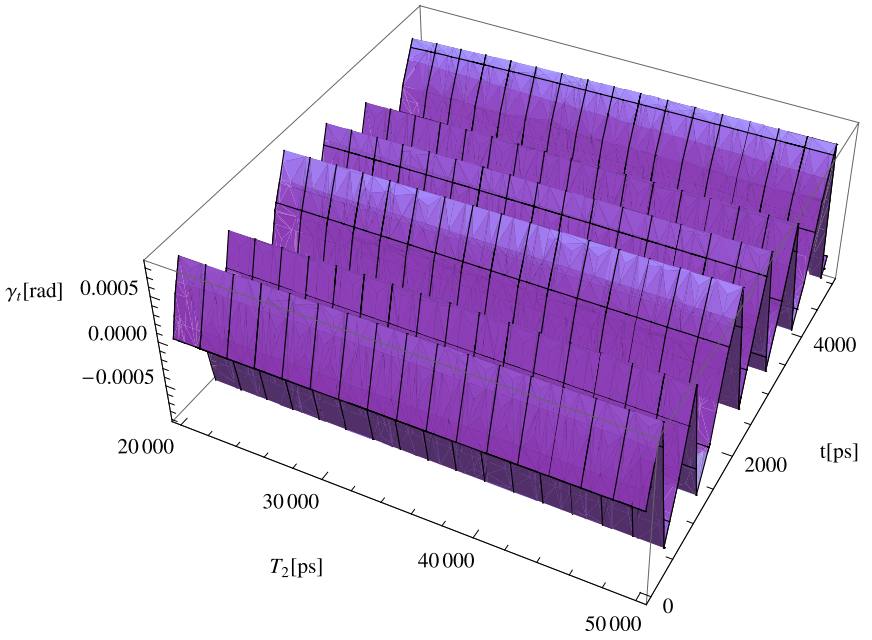


**Fig. 3** Same as Fig. 2 for total phase in optical free induction decay

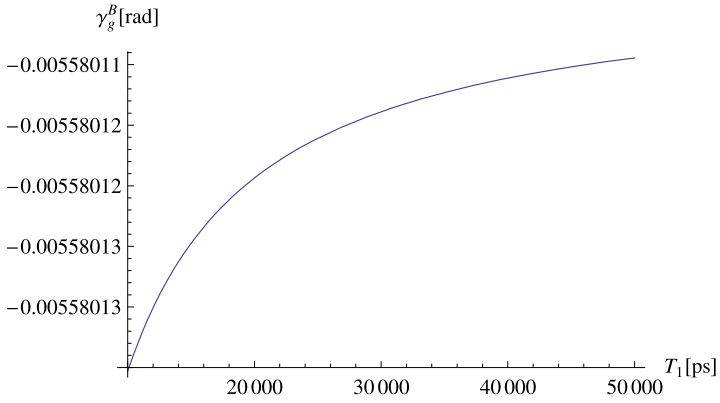


**Fig. 4** Berry phase with the quasicyclicity $T = \frac{2\pi}{\Delta'}$ for optical free induction decay is as a function of longitudinal relaxation time $T_1$ with parameters $\chi = 2.1 \times 10^8$, $w^0 = 0.1$, $\Delta' = 3.6 \times 10^9$ rad/s and $T_2 = 60 \times 10^{-9}$ s. The results show that the Berry phase increases with the longitudinal relaxant time

## 4 Geometric Phases as a Degree of Freedom of the Population Inversion and Heterodyne Beat Signal

Geometric phases are important both in a fundamental point of physical view and for their applications. Moreover, the physical system retains a memory of its evolution in terms of the geometric phase, which makes that the geometric phase has observable effect. There-

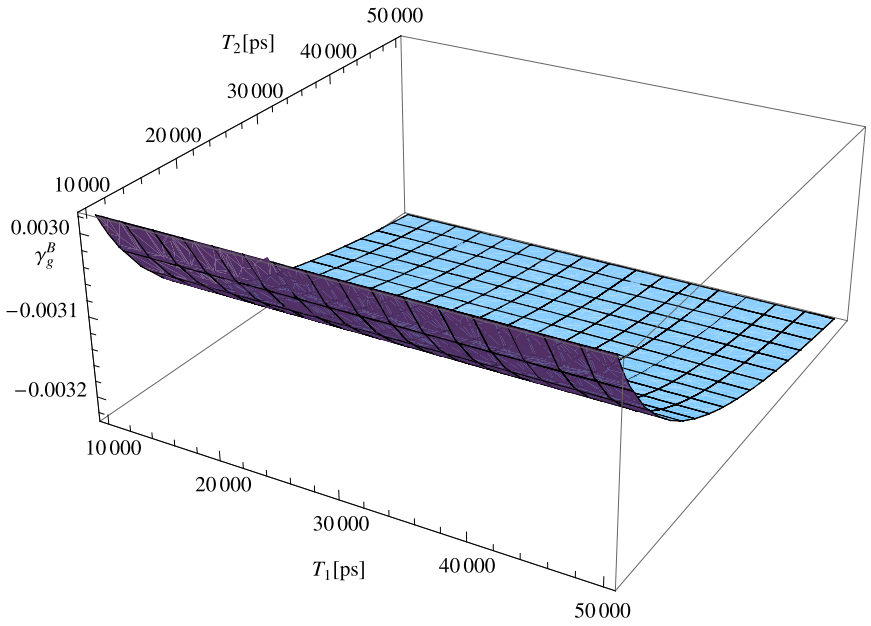


**Fig. 5** Three dimensional plot for the Berry phase with the quasicyclicity $T = \frac{2\pi}{\Delta'}$ is as a function of the transverse $T_2$ and longitudinal $T_1$ relaxation times with parameters $\chi = 2.1 \times 10^8$, $w^0 = 0.1$ and $\Delta' = 3.6 \times 10^9$ rad/s, where the oscillation disappears because it is accumulated at the region $t \in [0, T = 2\pi/\Delta']$. The results show that the Berry phase decreases with the transverse relaxation time

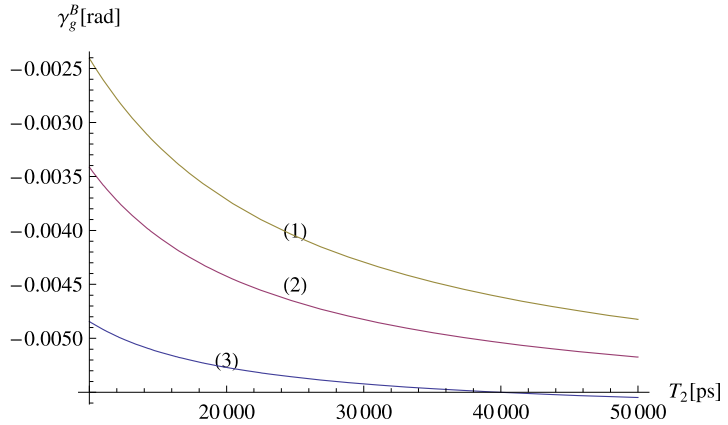


**Fig. 6** Berry phase is as a function of transverse relaxation time for the different regions of evolving time (*1*) $t \in [0, T = \frac{2\pi}{\Delta'}]$, (*2*) $t \in [T = \frac{2\pi}{\Delta'}, 2T = \frac{4\pi}{\Delta'}]$, (*3*) $t \in [2T = \frac{4\pi}{\Delta'}, 3T = \frac{6\pi}{\Delta'}]$ with parameters $\chi = 2.1 \times 10^8$, $w^0 = 0.1$, $\Delta' = 3.6 \times 10^9$ rad/s and $T_1 = 30 \times 10^{-9}$ s. The results imply that an exactly cyclic motion does not exist for the optical free induction decay

fore, it is very interesting in studying the relations between the population inversion and the geometric phase and between the heterodyne beat signal and the geometric phase.

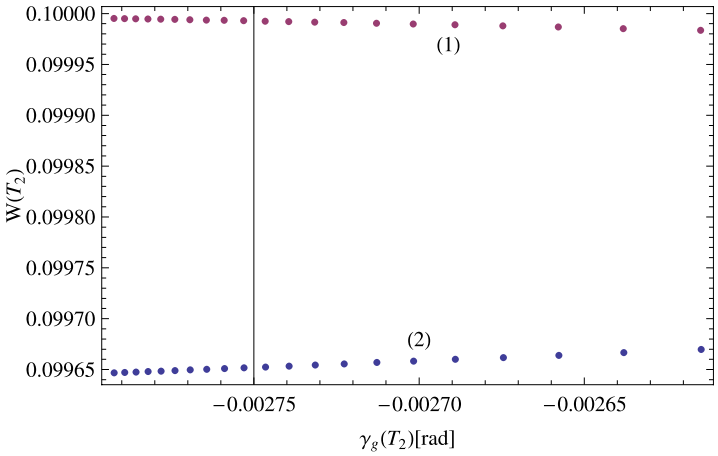


**Fig. 7** Population inversion $W = \rho_{aa} - \rho_{bb}$ in the process of optical free induction decay is as a function of geometric phase at evolving time $t = 10^{-9}$ s with different relations (*1*) $T_1 = T_2/(\gamma_h T_2 - 1)$ with $\gamma_h = 10^9$/s, (*2*) $T_1 = T_2$ for the parameters $\chi = 2.1 \times 10^8$, $w^0 = 0.1$ and $\Delta' = 3.6 \times 10^9$ rad/s

The relation between the population inversion and the geometric phase in the process of optical free induction decay is shown at the parametric Fig. 7. It is obvious that the population inversion is a liner function of the geometric phase. Moreover, the linear function depends on the relaxant times. Figure 7 shows that it is a linearly decreasing function for the relation $T_1 = T_2/(\gamma_h T_2 - 1)$ with $\gamma_h = 10^9$/s and a linearly increasing function for the relation $T_1 = T_2$. The linear relations mean that it may be a better way to describe the population inversion in process of the optical free induction decay by the geometric phase and therefore conceptually useful to analyze the fault-tolerance associated to the experiment and computational errors.

Differently from the population inversion, we find that the normalized intensity of heterodyne beat signal is evolving with the Berry phase $\gamma_g^B$ as a linearly decreasing function for $T_1 = T_2/(\gamma_h T_2 - 1)$ with $\gamma_h = 10^9$/s and a linearly increasing function for $T_1 = T_2$, respectively (see Figs. 8 and 9). Moreover, it is different for the different regions of evolving time because there does not exist an exact cyclic evolution in the optical induction decay (see Fig. 10).

It is known that the evolving memory of quantum system is kept in terms of a geometric phase. While the geometric phase is the area spanned in parameter space, which means that the phase may be controlled by the external parameter space. By taking the geometric phase as a basic degree freedom, thus, it may be better to control the devices of quantum memory by using varying external parameters without the usual operational process and therefore helpful to keep temporal storage of quantum information in the devices of quantum memory. Especially, it may be implete a scheme to analyze the fault-tolerance associated to the experimental and computational errors in design of the devices of quantum memory.

## 5 Kinematic Approach to Berry Phase

In order to compare the kinematic approach [24] to the Berry phase of mixed state, we compute the eigenvalues and eigenfunctions of the reduced density matrix in terms of the

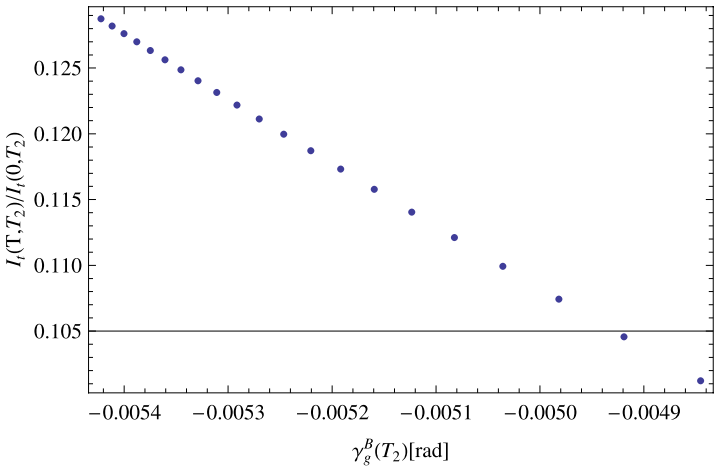


**Fig. 8** The normalized intensity of the heterodyne beat signal $I_t(T,T_2)/I_t(0,T_2)$ is as a function of Berry phase with the quasicyclicity $T=\frac{2\pi}{\Delta'}$ for the relation $T_1=T_2/(\gamma_h T_2-1)$ with $\gamma_h=10^9$/s for the parameters $\delta\omega_0=8.4\times10^8$ rad/s, $\chi=2.1\times10^8$, $w^0=0.1$ and $\Delta'=3.6\times10^9$ rad/s, which is a linearly decreasing function

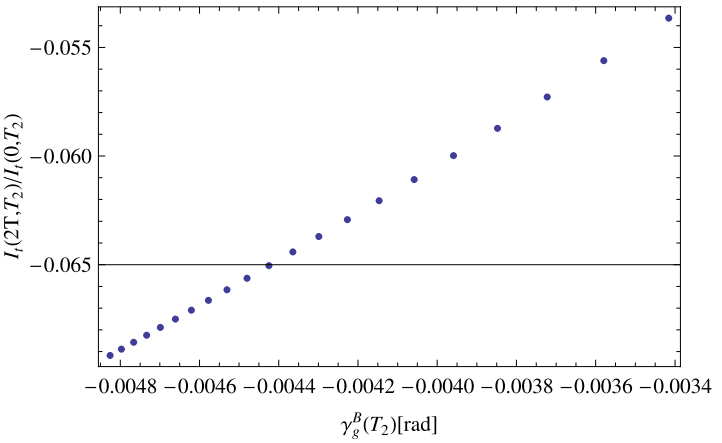


**Fig. 9** The normalized intensity of the heterodyne beat signal $I_t(2T,T_2)/I_t(0,T_2)$ is as a function of Berry phase at $t\in[T=\frac{2\pi}{\Delta'},2T=\frac{4\pi}{\Delta'}]$ with the relation $T_1=T_2$ for the parameters $\delta\omega_0=8.4\times10^8$ rad/s, $\chi=2.1\times10^8$, $w^0=0.1$, $\Delta'=3.6\times10^9$ rad/s, which is a linearly increasing function

Bloch vectors $u(t)$, $v(t)$ and $w(t)$. The eigenvalues may be expressed as

$$\lambda_1(t)=\frac{1}{2}(1-2\sqrt{u^2(t)+w^2(t)+v^2(t)}), \tag{28}$$

$$\lambda_2(t)=\frac{1}{2}(1+2\sqrt{u^2(t)+w^2(t)+v^2(t)}), \tag{29}$$

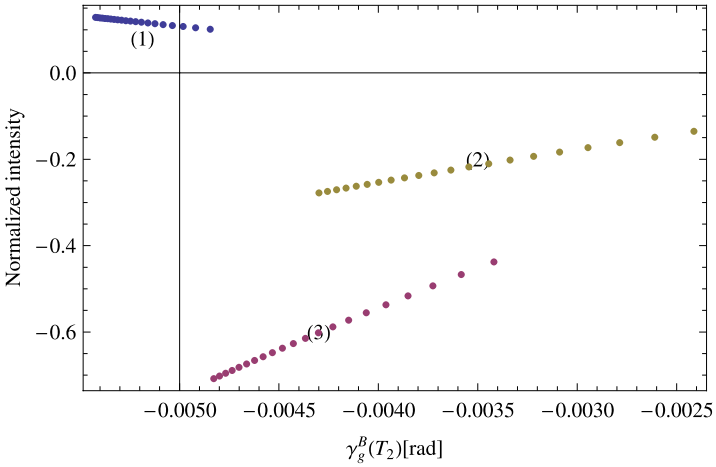


**Fig. 10** The normalized intensity $I_t(t,T_2)/I_t(0,T_2)$ of the heterodyne beat signal is as a function of Berry phase at (*1*) $t\in[0,T=\frac{2\pi}{\Delta'}]$, (*2*) $t\in[T=\frac{2\pi}{\Delta'},2T=\frac{4\pi}{\Delta'}]$, (*3*) $t\in[2T=\frac{4\pi}{\Delta'},3T=\frac{6\pi}{\Delta'}]$ with the relation $T_1=T_2/(\gamma_h T_2-1)$ with $\gamma_h=10^9$/s, $\delta\omega_0=8.4\times10^8$ rad/s, $\chi=2.1\times10^8$, $w^0=0.1$ and $\Delta'=3.6\times10^9$ rad/s. The results show that, for the different regions, the linear relations are different

and the corresponding eigenvectors are given by

$$|\lambda_1(t)\rangle=\begin{pmatrix} w(t)-\sqrt{u^2(t)+w^2(t)+v^2(t)} \\ u(t)+iv(t)\end{pmatrix}, \tag{30}$$

$$|\lambda_2(t)\rangle=\begin{pmatrix} w(t)+\sqrt{u^2(t)+w^2(t)+v^2(t)} \\ u(t)+iv(t)\end{pmatrix}. \tag{31}$$

According to the kinematic approach in Ref. [24], the eigenfunctions were directly taken as the normalized eigenvectors. Under this case, the kinematic approach to geometric phase has the significant difference of one quantity order in comparison with our approach. For the system with the nonunitary evolution, fortunately, the normalization eigenvectors used in the geometric phase are not arbitrary as discussed in Refs. [52, 53]. As pointed out by Anandan [53], the suitable form for the geometric phase from the kinematic approach under the nonunitary evolution should be modified as

$$\gamma_g^K=\arg\left(\sum_k\sqrt{\lambda_k(0)\lambda_k(T)}\langle\lambda_k(0)|\lambda_k(T)\rangle\exp\left(-\int_0^T dt\frac{\langle\lambda_k(t)|\frac{d}{dt}|\lambda_k(t)\rangle}{\langle\lambda_k(t)|\lambda_k(t)\rangle}\right)\right). \tag{32}$$

By analyzing the numerical calculations, we find that, through the results between the modified kinematic approach and our one are not fully agreement, the differences are very small for the geometric phase in process of the optical free induction decay. By comparing Fig. 4 with Fig. 11 with the same parameters for $T_1=T_2$, one may see that both are almost agreement. Similar situations are for $T_1=T_2/(\gamma_h T_2-1)$ with $\gamma_h=10^9$/s, which is shown at Fig. 12. Therefore, we conclude that our approach to the Berry phase of mixed state is very close to the modified kinematic one under the approximation of quasicyclic evolution.

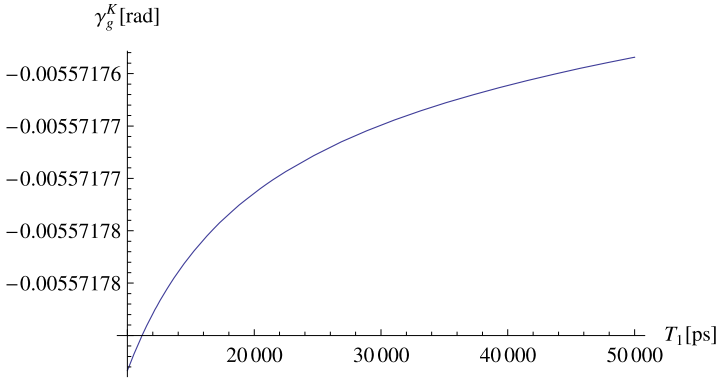


**Fig. 11** Berry phase from the modified kinematic approach with the quasicyclicity $T = \frac{2\pi}{\Delta'}$ is as a function of longitudinal relaxation time $T_1$ with parameters $\chi = 2.1 \times 10^8$, $w^0 = 0.1$, $\Delta' = 3.6 \times 10^9$ rad/s and $T_2 = 60 \times 10^{-9}$ s. The results show that the geometric phase increases with the longitudinal relaxation time. Comparing Fig. 4 from our approach with Fig. 11, we find that the difference of both approaches is very small

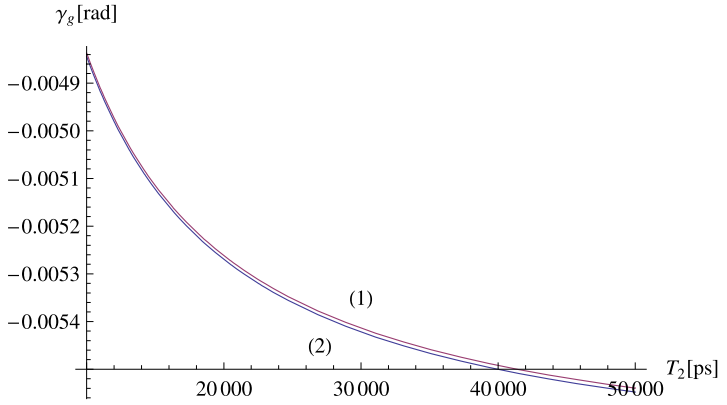


**Fig. 12** Geometric (Berry) phase with the quasicyclicity $T = \frac{2\pi}{\Delta'}$ is as a function of transverse relaxation time $T_2$ with relation $T_1 = T_2$, $\chi = 2.1 \times 10^8$, $w^0 = 0.1$, $\Delta' = 3.6 \times 10^9$ rad/s for the different approaches: (*1*) the kinematic one and (*2*) our one. The results show that, after a suitable modification to the kinematic one, our approach is very close to the modified kinematic one

In the kinematic approach to the geometric phase [24], however, the state vector of the mixed state was taken as a purified state. It is well-known that the purification of the density operator is not usually unique. Under the limiting of pure state, the kinematic approach can not give out the Berry phase and Aharonov and Anandan one. In contrast to the kinematic approach, our approach include the results of Berry phase and Aharonov and Anandan one in the case of pure state. In addition, the kinematic approach only consider the system with the quasicyclic evolution that does not exist in the really physical system, which can not avoids the problems associated with types of errors that do not preserve cyclicity and therefore may not be conceptually useful in that it makes it possible to analyze the fault tolerance associated to such errors.

## 6 Conclusions

In summary, the geometric phase for the open system with a modified Bloch equation is calculated by establishing in connecting density matrices, describing an evolution of the optical free induce decay, with the nonunit vector ray in the complex projective Hilbert space. Because the geometric phase depends only on the smooth curve on this Bloch parameter space, it is formulated entirely in terms of geometric structures. We show that, furthermore, our approach to the geometric phase of mixed state may give out the Berry phase and Aharonov and Anandan one under the limiting of the pure state. By comparing our approach with the kinematic one, a suitable modification is proposed for the kinematic approach to the mixed state. Then, we find that our geometric phase of mixed state under the quasicyclic approximation may include the results of the modified kinematic one.

The relations between the geometric phase and population inversion and between the geometric phase and the heterodyne beat signal field are analyzed. We find that both the population inversion and the normalized heterodyne beat signal intensity are a linear function of geometric phase, which depend strongly on the relaxant times and the relations between the transverse and longitudinal relaxant ones. The results may be helpful for searching for devices of quantum memory by the photon echo.

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